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THEORY OF LIFTING SURFACES.

PART II.

by

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# THEORY OF LIFTING SURFACES

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## PART II.

### A. THEORY OF MULTIPLANES.

The determination of the resistance  $W_{12}$  due to the influence of a wing 2 on a wing 1, is obtained by calculating by means of equation (14) the speed  $W_{12}$ , due to wing 2 at a point situated in the same plane as the lifting screw, this plane being perpendicular to the direction of the stream. We obtain:

$$W_{12} = \frac{1}{4\pi} \int_0^{l_1} \int_0^{l_2} \frac{\Gamma_1 \Gamma_2 \cos(\beta_1 + \beta_2) ds_1 ds_2}{a^2} \quad (30)$$

where  $l_1$  is the span of the wing;  $a$ , the distance from the point considered to the origin of the coordinates, and  $\beta$  the angle with one of the  $Z$  lines on the right which joins the point considered to the origin of the axes.

The expression (30), being symmetrical in relation to wings 1 and 2, shows that the resistance to which wing 2 is subjected on account of the presence of wing 1, is the same as that which has just been determined. Otherwise stated:

$$W_{12} = W_{21}$$

This relation, which is of very great importance, was first found by means of another method by Mr. Munk, one of the author's collaborators.

It may be formulated in a more general way as follows:

If in a lifting system, the elements of which are situated in the same plane, we choose two groups, the resistance of group 1 due to group 2, is the same as the resistance of group 2 due to group 1.

The determination of resistance in the more general case where the elements are not in the same plane perpendicular to the stream, leads to the conclusion that the resistances induced by one wing on the other are only equal when, as assumed above, the wings are in the same plane perpendicular to the direction of the stream. But if the wings are staggered, the sum of resistances remains constant whatever be the amount of stagger.

It is however necessary that, according to the amount of stagger, the angles of attack of the wings are such that the lift is always the same. The sum of the resistances induced by one wing on the other is given by the formula:

$$W_{12} + W_{21} = \frac{\rho}{2\pi} \iint \frac{\Gamma_1 \Gamma_2}{a^2} \cdot \cos(\beta_1 + \beta_2) ds_1 ds_2 \quad (33)$$

Formulas (33) and (14) show that total resistance is a well defined function of the lifts  $A_1, A_2, \dots, A_n$ .

In an article entitled "Induced Resistance of Multiplanes" which appeared in the "Technischen Berichten der Flugzeugmeisterei" Vol. III, p. 309, the author shows that the total induced resistance  $W$  of a biplane, the wings of which had spans equal to  $b_1$  and  $b_2$ , the difference being equal to  $b_1$ , can be expressed by the formula:

$$W = \frac{1}{\pi q b_1^2} (A_1^2 + 2 \delta \mu A_1 A_2 + \mu^2 A_2^2) \quad (36)$$

where  $\delta$  is a certain function of  $2h/(b_1 + b_2)$  and of  $\mu = b_2/b_1$ .

We can thus solve the question as to what should be the ratio of  $A_2$

to  $A_1, A_2$  &  $A_1$  being constant, for  $W$  to be minimum.

The necessary condition is:

$$A_2 : A_1 = (\mu - \delta) : \left( \frac{1}{\mu} - \delta \right);$$

We obtain:

$$W_{\min} = \frac{(A_1 + A_2)^2}{\pi a b_1^2} \frac{1 - \delta^2}{1 - 2\delta\mu + \mu^2} \quad (37)$$

which shows,  $\delta$  being  $< \mu$ , that this resistance is less than that of a monoplane of span  $b_1$ , lifting  $A_1 + A_2$ .

If we take the maximum span  $b_1$ , the minimum resistance will occur for  $b_2 = b_1$  and  $A_2 = A_1$ .

If we designate by  $\nu$  the ratio of the induced resistance of a lifting system to that of a monoplane having the same span and the same total lift, the induced resistances of the two lifting systems, having the same lift  $c_a$  are connected by the expression

$$c_{w1} - c_{w2} = \frac{c_a^2}{\pi} \left( \frac{\nu_1 F_1}{b_1^2} - \frac{\nu_2 F_2}{b_2^2} \right) \quad (38)$$

This formula is due to Mr. Munk.

## B. LIFTING SYSTEMS OF MINIMUM RESISTANCE.

The question of the minimum resistance of a lifting system of which the mean wing sections have given dimensions and supply a determined lift has been studied in a work by Mr. Munk, a collaborator of the author.

The question has again been taken up by the author himself under another form and he has been led to assume that the minimum resistance of a lifting system is

$$W = A^2 / 4qF'$$

where  $F'$  is a surface the dimensions of which depend on the condi-

tions of the problem.

Three cases are examined: the first refers to a biplane having the two wings of the same span. Taking  $V$  as the ratio of the induced resistance of this biplane to the resistance of a monoplane of the same span carrying the same load, and  $h/b$  the ratio of the gap to the span of the wings, we have

$$V \approx \frac{1 + 1.63 h/b}{1.027 + 3.84 h/b} \quad (44)$$

The second case is that of a biplane having the upper and lower planes connected by two vertical planes placed at the tips of the wings. This biplane gives less resistance than any of the multiplanes occupying the same space. We have:

$$V \approx \frac{1 + 0.45 h/b}{1.045 + 2.8 h/b} \quad (45)$$

The third case is that of a monoplane having a slit of width  $d$  in the middle of the wing. By means of this device the resistance is that of a wing having a span  $b - d$ .

#### C. FREE STREAM AND STREAM LIMITED BY WALLS.

As regards the value of test results, it is obviously of great importance to examine the question of the influence of the limitations imposed on the air-stream in which models are tested.

Two methods are generally employed:

- (a) The tunnel
- (b) A free stream traversing the experimental chamber.

The conditions at the limits are: for the first, the nullifying of the normal component,  $w_n$ , of the speed. For the second, the constancy of pressure.

This latter condition leads to the relation:

$$(V + v)^2 + u^2 + w^2 = \text{const.} = V^2$$

whence, neglecting the terms of secondary importance

$$v = 0.$$

These two problems may be solved by superposing on the flow of the fluid extended to infinity another flow to the potential of velocities, such that the velocities at the limits due to this potential have the values  $-w_n$  and  $-v$ .

The components of the velocities due to this potential perpendicular to the wing, modify the angle of attack of the wing in the same way as if these modifications arose from other wings.

The author has examined this question in detail for a single wing with elliptical distribution of lift placed in a Wind Tunnel of circular section having a free stream.

$$\begin{aligned} \text{Let } b & \text{ be the span of the wing;} \\ D & \equiv 2R, \text{ the diameter of the stream;} \\ \xi & = 2xb/D^2. \end{aligned}$$

The component of the speed normal to the direction of the stream, due to the conditions at the limits is

$$W' = \frac{A}{4\pi R^2 \rho v} \left( 1 + \frac{3}{4} \xi^2 + \frac{5}{8} \xi^4 + \frac{35}{128} \xi^6 + \dots \right) \quad (48)$$

The additional resistance due to this speed is:

$$W' = \frac{A^2}{4\pi R^2 \rho v^2} \left( 1 + \frac{3}{16} \left( \frac{b}{D} \right)^4 + \frac{5}{64} \left( \frac{b}{D} \right)^8 + \dots \right) \quad (49)$$

Keeping only the first term of this relation, we find that the total induced resistance is:

$$W_{*'} = \frac{A^2}{2\rho v^2} \left( \frac{1}{F'} + \frac{1}{2F_0} \right)$$

where  $F_0$  is the surface of the section of the air-stream.

For a tunnel with uninterrupted walls the correction is the same, but of contrary sign, that is, the resistance measured is smaller than the true resistance in an unlimited stream of air. This correction is far from being negligible: thus for  $b/D = 0.5$ , a usual case in laboratories, the additional resistance reaches about  $1/8$  of the induced resistances.

Another problem, that of a wing traversing a free cylindrical stream, is of practical interest on account of the simple installation required for the experiment.

The solution is given by the fact that the lift of the wing is nullified where the wing leaves the air-stream.

Assuming for serial development the function  $\frac{R^2 - x^2}{R^2 + x^2}$  we find that

$$W = 1.74 A^2/2 \rho v^2 F_0$$

that is, the measured resistance is greater by 1.74 than the resistance of the same wing in an unlimited stream.

Finally the author examines the case where the chord of the wing is such that the variation of the speed  $w_n$  parallel to the general direction of the stream can no longer be neglected, this leading to a modification of the camber of the wing.

The author considers more particularly the conditions of tests of a wing limited by two vertical planes (connecting the tips of the upper and lower wings) traversing the stream, the object being to determine the conditions of a plane-parallel flow.

The measured resistance is then

$$W = A/2 \rho v^2 F_0$$

where  $F_0$  is still the section of the stream.

Also the camber of the wing should be increased by the value

$$1/R = \pi c_a t / 12 h^2$$

(h being the height of the stream) in order that the test shall give the same results as that of the wing with the original camber and of infinite span in an infinite stream.

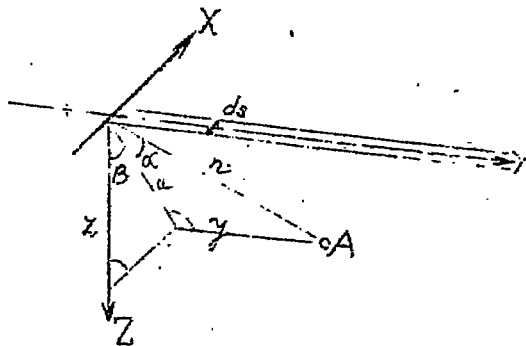
In a tunnel with walls, the measured resistance is equal to the true resistance, but the camber must be diminished by half of the value given by the expression (58).

#### D. CONDITIONS OF FLOW AT A GREAT DISTANCE FROM THE WING.

The pressure at any point whatever in space outside the vortexes accompanying the wings is given by the relation

$$p - p_0 = \frac{\rho v}{4\pi} \int T \, dx \cdot \frac{\cos \alpha \cos \beta}{r^2} = \frac{Az}{4\pi r^3} \quad (60)$$

The various notations comprised in this formula are explained by the following sketch:



A super-pressure reigns beneath the lifting system, and a depression above it.

The integral of the difference of the pressures borne by an infinite surface perpendicular to the Z arc is  $\pm \frac{A}{2}$  according as to whether the surface is above or below the lifting system. Two of these surfaces surrounding a lifting system thus bear the whole



of the lift under form of pressure:

If one of these surfaces is at a finite distance from the lifting system, as is the case for an airplane flying above the surface of the earth, it will be necessary, in order to satisfy the conditions at the limits, to assume, below the surface of the earth, an analogous lifting system directed in such a way as to give a direct image of the first system.

The pressure on the surface of the earth will consequently be equal to  $A_1$  while the pressure on each surface above the airplane will be null.

This pressure is transported instantaneously - or, if we take into account the compressibility of the air - with the velocity of sound, on the surface of the earth.

In examining the conditions of the flow of the fluid in the vortexes which take their rise at the tips of the wings, the author determines the velocity of these vortexes in the case of a monoplane and of a biplane and their relations with the lift and resistance of the lifting system.

In terminating, the author indicates the various applications which the theories exposed admit of. He specially remarks the development of the theory of the lifting-screw as a theory of the lifting surface which would, on the one hand, enable us to determine the influence of the profile on lift and on the distribution of lift, and on the other hand would help us to study the influence of wing camber in biplanes better than has hitherto been possible.

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Another application would consist in studying oblique surfaces curved in the direction of the movement.

The influence of a variation of lift in function of the time, as occurs in flapping wings, might also be studied.

The author finally considers applications to flight on curved trajectories, to propellers, fans, and turbines.

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